

# Quantum Information Theory

Final exam  
Spring term 2025

From: 4th July 2025, 09h15  
To: 4th July 2025, 12h15

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## PHYS-550 – Exam – room BCH 2201

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section. **Please mark clearly which two you have answered below.** (If you don't follow this instruction (e.g. you mark all the 3 questions below, or you don't mark any of them below), then only long Problems A and B will be graded.)
- **Please start a new sheet for each of the long answer questions**, or a penalty of 10 points will be deducted from your total number of points.
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short problems:	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
<b>Total</b>	/100

# Shorter (easier) questions

Please answer all of the following questions.

## 1. States

Let  $\rho^{AB}$  be a density operator on the composite Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and let

$$\rho^A = \text{Tr}_B[\rho^{AB}] \quad (1)$$

be its reduced state on subsystem  $A$ .

Consider a projective measurement on  $A$  given by the orthogonal projectors  $\{\Pi_i^A\}$ , with  $\sum_i \Pi_i^A = I^A$ .

- (a) Show that the probability of outcome  $i$  when measuring  $\{\Pi_i^A\}$  on the joint state  $\rho^{AB}$ ,

$$p_i = \text{Tr}\left[(\Pi_i^A \otimes I^B) \rho^{AB}\right], \quad (2)$$

can be written purely in terms of the reduced state  $\rho^A$ , namely

$$p_i = \text{Tr}[\Pi_i^A \rho^A]. \quad (3)$$

(4 marks)

- (b) Explain what is meant by ‘purifying’ a mixed state  $\rho$ .

(2 marks)

- (c) Show that it is possible to purify any state  $\rho$ . How large a purifying system is required to do so?

(4 marks)

## 2. Measurements

Consider the following three-outcome POVM on a qubit:

$$E_i = \frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma}), \quad i = 1, 2, 3, \quad (4)$$

where  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the Pauli matrices.

- (a) What constraints do we need on the  $\mathbf{n}_i$  vectors to ensure that this is a valid POVM?

(6 marks)

- (b) Suppose we have a state  $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$  where  $\|\mathbf{r}\| \leq 1$ . Show that the probability of measuring outcome  $i$  is given by

$$p_i = \frac{1}{3}(1 + \mathbf{n}_i \cdot \mathbf{r}). \quad (5)$$

(2 marks)

- (c) Specify a choice in the vectors  $\mathbf{n}_i$  such that each POVM element is proportional to a projective measurement.

Describe what measurement procedure this case corresponds to.

(2 marks)

### 3. Shot noise

Consider the observable  $O = 3\sigma_x + 2\sigma_z$  and the initial state  $\rho = |0\rangle\langle 0|$ . Suppose we measure this observable by first rotating  $\rho$  into the eigenbasis of  $O$ . Bound how many shots  $N$  suffices to estimate  $\text{Tr}[\rho O]$  to precision  $\epsilon$  with probability  $1 - \delta$ :

- (a) Using Chebyshev's inequality

(4 marks)

- (b) Using Hoeffding's inequality.

(6 marks)

### 4. Entropies

Show that a pure state  $|\psi\rangle_{AB}$  is entangled between subsystems  $A$  and  $B$  if and only if the conditional entropy is less than zero i.e.,  $S(A|B) < 0$ .

(10 marks)

### 5. Entanglement

Consider the two-qubit states  $\rho_p$  defined by

$$\rho_p = p |\Psi^-\rangle\langle\Psi^-| + (1-p) \frac{I}{4}, \quad (6)$$

where  $I$  is the  $4 \times 4$  identity,  $p \in (0, 1)$ , and

$$|\Psi^-\rangle = \frac{|0, 1\rangle - |1, 0\rangle}{\sqrt{2}} \quad (7)$$

is the singlet (maximally entangled) state.

Determine the range of  $p$  for which  $\rho_p$  is separable and for which it is entangled.

(10 marks)

# Longer (harder) questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

Hints are only hints - you may find an alternative approach that works for you.

## Question A - Fidelity

The goal of this problem is to derive, and use, the following alternative expression for the fidelity:

$$F(P, Q) := \text{Tr} \left[ \sqrt{P^{\frac{1}{2}} Q P^{\frac{1}{2}}} \right] = \text{Tr} \left[ \sqrt{PQ} \right], \quad (8)$$

which holds if  $P$  is positive definite.

Let matrices  $A, B > 0$ , we call

$$A \# B := A^{\frac{1}{2}} \sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} A^{\frac{1}{2}} \quad (9)$$

the geometric mean between  $A$  and  $B$ .

- (a) Show that the geometric mean  $X = A \# B$  is the unique positive solution of the Riccati equation  $XA^{-1}X = B$ .

*Hint: You can first show that the geometric mean solves this equation, and then show the uniqueness.*

(9 marks)

- (b) Next, let  $P, Q$  be two positive definite density matrices. Show that

$$\text{Tr} \left[ \sqrt{PQ} \right] = \text{Tr} \left[ \sqrt{P^{\frac{1}{2}} Q P^{\frac{1}{2}}} \right]. \quad (10)$$

*Hint: The Riccati equation will be helpful if you find the right way to use it.*

(9 marks)

- (c) Hence, or otherwise, compute the fidelity between the following states:

i)  $\rho = \frac{1}{2}(I + \sigma_x)$  and  $\sigma = \frac{1}{2}(I + \sigma_y)$

ii)  $\rho = \frac{1}{2}(I + 0.5\sigma_x)$  and  $\sigma = \frac{1}{2}(I + 0.8\sigma_x)$

iii)  $\rho = \frac{1}{2}(|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|)$  and  $\sigma = \frac{1}{2}(|\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|)$

(7 marks)

## Question B - Majorization and entanglement

Let  $\mathbf{x} = [x_1, x_2, \dots, x_d] \in \mathbb{R}^d$  and  $\mathbf{y} = [y_1, y_2, \dots, y_d] \in \mathbb{R}^d$  be two vectors such that  $x_1 \geq x_2 \geq \dots \geq x_d$  and  $y_1 \geq y_2 \geq \dots \geq y_d$ . We say that  $\mathbf{x}$  is *majorized* by  $\mathbf{y}$  if

$$\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i \quad \forall j \in \{1, 2, \dots, d-1\}, \quad (11)$$

$$\sum_{i=1}^d x_i = \sum_{i=1}^d y_i. \quad (12)$$

Equivalently,  $\mathbf{x}$  is majorized by  $\mathbf{y}$  if and only if there exists a doubly stochastic matrix  $D$  such that  $D\mathbf{y} = \mathbf{x}$ . (A doubly stochastic matrix is one such that  $\sum_i D_{ij} = 1$  for all  $j$  and  $\sum_j D_{ij} = 1$  for all  $i$ .)

Consider two  $d$ -dimensional density matrices  $\rho, \sigma \in \mathbb{C}^d$  such that

$$\rho = \sum_{i=1}^n p_i U_i \sigma U_i^\dagger, \quad (13)$$

where  $U_1, U_2, \dots, U_n$  are unitary and  $\sum_{i=1}^n p_i = 1$ .

- (a) Show that the eigenvalues of  $\rho$  are majorized by the eigenvalues of  $\sigma$ .  
(6 marks)
- (b) Suppose that  $B, C$  are  $n$  dimensional complex matrices such that  $B^\dagger B = C^\dagger C$ . Then show that there must exist a unitary  $U \in \mathbb{C}^{n \times n}$  such that  $B = UC$ .  
(4 marks)

Consider now a bipartite pure state  $|\psi^{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  between Alice and Bob. Alice and Bob perform the following LOCC operation:

- i. Alice performs a POVM with Kraus operators  $M_1, M_2, \dots, M_n$ .
- ii. Alice sends the measurement outcome to Bob.
- iii. Depending on the measurement outcome  $k \in \{1, 2, \dots, n\}$ , Bob applies a channel  $\mathcal{E}_k$  on his state.

Suppose that the measurement operators  $M_k$  and channels  $\mathcal{E}_k$  are chosen such that after this protocol Alice and Bob still share a pure state  $|\phi^{AB}\rangle$ .

The output of this protocol is thus:

$$|\phi^{AB}\rangle\langle\phi^{AB}| = \sum_i (I \otimes \mathcal{E}_i)((I \otimes M_i)|\psi^{AB}\rangle\langle\psi^{AB}|(I \otimes M_i^\dagger)) \quad (14)$$

(c) If  $\rho_\psi = \text{Tr}_B(|\psi^{AB}\rangle\langle\psi^{AB}|)$  and  $\rho_\phi = \text{Tr}_B(|\phi^{AB}\rangle\langle\phi^{AB}|)$ , show that  $\forall i \in \{1, 2, \dots, k\}$ ,

$$M_i \rho_\psi M_i^\dagger \propto \rho_\phi. \quad (15)$$

*Hint- given that the output state of the LOCC protocol is **pure**, what does this imply about each of the terms  $(I \otimes \mathcal{E}_i)((I \otimes M_i)|\psi^{AB}\rangle\langle\psi^{AB}|(I \otimes M_i^\dagger))$  ?*

(7 marks)

(d) Show that the Schmidt coefficients of  $|\psi^{AB}\rangle$  are majorized by those of  $|\phi^{AB}\rangle$ .

*Hint - use (b) and (c) to express  $\rho_\psi$  as a convex combination of unitaries drawn from a probability distribution and applied on  $\rho_\phi$ .*

(6 marks)

(e) What theorem in entanglement theory is your answer to (d) an example of? Is it consistent or inconsistent with that theorem? Explain your reasoning.

(2 marks)

### Question C - Vectorization

Quantum information theorists love computing averages over random unitaries.

To do so, we often make use of the following standard identities:

$$\int_{U(d)} U \otimes U^* dU = \frac{1}{d} |\text{vec}(I_d)\rangle\langle\text{vec}(I_d)| \quad (16)$$

and

$$\begin{aligned} & \int_{U(d)} U^{\otimes 2} \otimes (U^*)^{\otimes 2} dU \\ &= \frac{1}{d^2 - 1} \left( |\text{vec}(I_{d^2})\rangle\langle\text{vec}(I_{d^2})| - \frac{1}{d} |\text{vec}(I_{d^2})\rangle\langle\text{vec}(F)| - \frac{1}{d} |\text{vec}(F)\rangle\langle\text{vec}(I_{d^2})| + |\text{vec}(F)\rangle\langle\text{vec}(F)| \right) \end{aligned} \quad (17)$$

where  $d$  is the dimension of  $U$ ,  $U(d)$  is the uniform ensemble of all unitaries of dimension  $d$ ,  $I_d$  is the identity of dimension  $d$ , and  $F := \sum_{i,j=1}^d |i,j\rangle\langle j,i|$  is the SWAP operator.

*Remark: You do not need to understand where these identities come from, or even fully understand them, to answer this question! (But if you've previously studied group averaging maybe you can see how they would be derived.)*

This question will walk you through how to use vectorization to use these identities.

Let us start by supposing that given some initial state  $\rho$  and observable  $O$  we are interested in computing the expectation value

$$\int_{U(d)} \text{Tr}[U\rho U^\dagger O] dU. \quad (18)$$

(a) Show that

$$\text{Tr}[U\rho U^\dagger O] = \langle\text{vec}(O)|U \otimes U^*|\text{vec}(\rho)\rangle \quad (19)$$

(1 mark)

(b) Hence use Eq. (16) to compute  $\int_{U(d)} \text{Tr}[U\rho U^\dagger O] dU$ . What is its value if  $O$  is a Pauli matrix?

(3 marks)

(c) Now show that  $\text{Tr}[U\rho U^\dagger O]^2 = \langle\text{vec}(O^{\otimes 2})|U^{\otimes 2} \otimes U^{*\otimes 2}|\text{vec}(\rho^{\otimes 2})\rangle$ .

(2 marks)

(d) Hence use Eq. (17) to compute  $\int_{U(d)} \text{Tr}[U\rho U^\dagger O]^2 dU$ . What is its value if  $O$  is a Pauli matrix and  $\rho$  is a pure state?

(4 marks)

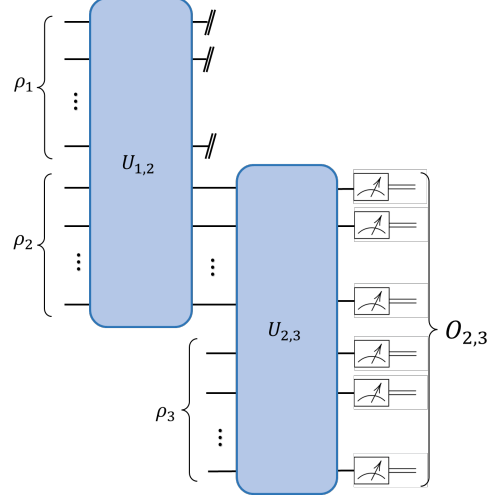


Figure 1: Illustration of the quantity  $X$  and the tripartite system.

Let's now consider a tripartite system consisting of three subspaces of dimension  $d$ . As shown in Fig. 1, we consider the quantity

$$X := \text{Tr}[O_{2,3}U_{2,3}(\sigma_2 \otimes \rho_3)U_{2,3}^\dagger] \quad (20)$$

where

$$\sigma_2 = \text{Tr}_1[U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger], \quad (21)$$

$O$  and  $U$ 's have dimension  $d^2$ , the density operators  $\rho$ 's and  $\sigma$  have dimension  $d$ , and we specify the subspace on which they act in their subscripts.

(e) Show that

$$X = \text{Tr}[(I_1 \otimes O_{2,3})(I_1 \otimes U_{2,3})(U_{1,2} \otimes I_3)(\rho_1 \otimes \rho_2 \otimes \rho_3)(U_{1,2} \otimes I_3)^\dagger(I_1 \otimes U_{2,3})^\dagger]. \quad (22)$$

(3 marks)

(f) Use vectorization to show that  $X$  can be written as

$$X = \sum_{Y \in \mathcal{B}} \left( \langle \text{vec}(I_1 \otimes Y) | U_{1,2} \otimes U_{1,2}^* | \text{vec}(\rho_1 \otimes \rho_2) \rangle \langle \text{vec}(O_{2,3}) | U_{2,3} \otimes U_{2,3}^* | \text{vec}(Y \otimes \rho_3) \rangle \right) \quad (23)$$

where  $\mathcal{B}$  is a set of operators. Specify a set of operators  $\mathcal{B}$  for which the equality holds.

*Hint: a resolution of identity may be helpful if you can figure out when to use it.*

(8 marks)

(g) We let  $U_{1,2} = U_{3,2} = U$ . Under this assumption, use Eq. (17) to compute  $\int_{U(d)} X dU$ .

(4 marks)